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Time Asymmetry and Quantum Theory of Resonances and Decay

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These notes review a consistent and exact theory of quantum resonances and decay. Such a theory does not exist in the framework of traditional quantum mechanics and Dirac's formulation. But most of its ingredients have been familiar entities, like the Gamow vectors, the Lippmann–Schwinger (in- and out-plane wave) kets, the Breit–Wigner (Lorentzian) resonance amplitude, the analytically continued S-matrix, and its resonance poles. However, there are inconsistencies and problems with these ingredients: exponential catastrophe, deviations from the exponential law, causality, and recently the ambiguity of the mass and width definition for relativistic resonances. To overcome these problems the above entities will be appropriately defined (as mathematical idealizations). For this purpose we change just one axiom (Hilbert space and/or asymptotic completeness) to a new axiom which distinguishes between (in-)states and (out)observables using Hardy spaces. Then we obtain a consistent quantum theory of scattering and decay which has the Weisskopf–Wigner methods of standard textbooks as an approximation. But it also leads to time-asymmetric semigroup evolution in place of the usual, reversible, unitary group evolution. This, however, can be interpreted as causality for the Born probabilities. Thus we obtain a theoretical framework for the resonance and decay phenomena which is a natural extension of traditional quantum mechanics and possesses the same arrow-of-time as classical electrodynamics. When extended to the relativistic domain, it provides an unambiguous definition for the mass and width of the Z-boson and other relativistic resonances.

KEY WORDS: time asymmetry; quantum theory; decay.

1. INTRODUCTION

Time Asymmetric Quantum Theory (TAQT) differs in its basic hypothesis very little from the traditional axioms of quantum mechanics and relativistic quantum field theory. The mathematical tools are linear operators in linear scalear product spaces, in general infinite dimensional ones. The physical quantities measured are the Born probabilities P for an observable (operator) A in a state (operator) W

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with the simplest case given by $A = |\psi\rangle\langle\psi|$, $W = |\phi\rangle\langle\phi|$, and

$$P(t) = |\langle\psi|\phi(t)\rangle|^2 = |\langle\psi(t)|\phi\rangle|^2. \tag{1}$$

The dynamical evolution is described by the Schrödinger equation for $\phi(t)$ or by the Heisenberg equation for $\psi(t)$. For infinite dimensional spaces one needs completeness (i.e., converging sequences have limit elements in the space) and for convergence one uses in traditional quantum mechanics the convergence with respect to the norm and then obtains the Hilbert space $\mathcal{H}$.

Though not part of the traditional axioms these basic hypotheses are usually augmented by the Dirac kets, e.g., of the Hamiltonian $|E\rangle$, and Dirac's basis vector expansion

$$\phi = \sum_{j, j_3, \eta} \int dE |E, j, j_3, \eta\rangle\langle E, j, j_3, \eta|\phi\rangle = \int |E\rangle\langle E|\phi\rangle. \tag{2}$$

The j, j_3, η are discrete quantum numbers which we often ignore. Further one makes some assumptions about the analyticity of the S-matrix element $S_j^{\eta\eta'}(E)$ as a function of the complex energy, which have been conjectured from solutions of the Schrödinger equation for certain potentials.

For stationary systems (e.g., atoms, nuclei, relativistic particles for which all *excited states are considered as stable*) the traditional assumptions give an adequate description. Augmenting these by the theory of (tempered) distributions, which extends the Hilbert space $\mathcal{H}$ to a Gelfand triplet or Rigged Hilbert Space (RHS),

$$\Phi \subset \mathcal{H} \subset \Phi^{\times}, \tag{3}$$

where Φ is the Schwartz space, one can give a mathematical meaning to Dirac kets: $|E\rangle \in \Psi^{\times}$, i.e., as functionals on the Schwartz space. Then the nuclear spectral theorem provides the mathematical proof for the Dirac's basis vector expansion (2). The traditional axiom, that the set of the physical states ϕ and that of observables ψ are given by $\mathcal{H}$ or by Φ, leads to the theory of stable states and reversible (*unitary*) time evolution. In contrast, quasistable states, like resonances in a scattering experiment or decaying states in the decay experiment, are connected with an asymmetric or "irreversible" time evolution (The irreversible nature of quantum mechanical decay has been mentioned in textbooks and lecture notes. For example, Antoniou, 1992; Antoniou and Prigogine, 1993; Cohen-Tannaudji, 1977; Doebner *et al.*, 1992; Goldberger and Watson, 1964; Haag, 1990, 1997; Lee, 1981; Merzabacher, 1970; van Kamper, 2002). Thus they require a TAQT and in the absence of such a theory their description can only be approximate and must contain some contradictions.

2. WIDTH AND LIFETIME

Resonances appear in scattering experiments when the scattering cross section, $\sigma_j(E)$, is fit to a Breit–Wigner energy distribution and a slowly varying background $B(E)$

$$\sigma_j(E) \sim |a_j(E)|^2, \quad a_j = a_j^{\mathrm{BW}} + B(E) = \frac{r_\eta}{E - \left(E_R - i\frac{\Gamma}{2}\right)} + B(E), \tag{4a}$$

where the Breit–Wigner (also called Lorentzian) is

$$a_j^{\mathrm{BW}} = \frac{r_\eta}{E - \left(E_R - i\frac{\Gamma}{2}\right)}; \qquad 0 \leq E < \infty \tag{4b}$$

and where η denotes the species quantum numbers of various final states (channels).

Resonances are characterized by the resonance energy E_R (or resonance mass M in the relativistic case) and by the resonance width Γ. The width Γ can be determined experimentally from (4a) when Γ/E_R (and likewise Γ/M) is of the order $10^{-1} \ldots 10^{-4}$.

Decaying states $\phi^D(t)$ are observed in process $D \to \eta$ where η are various decay products (or decay channels) described by the outvectors ψ_η. The decaying state D is characterized by $(E, 1/\tau \equiv R)$ (or by $(M, 1/\tau \equiv R)$) where τ is the lifetime (in the rest frame) and R is the total initial decay rate. The lifetime τ is measured by fitting the counting rate, $\frac{1}{N}\frac{\Delta N_\eta(t)}{\Delta t}$, for any decay product η to the exponential decay law for the partial decay rate $R_\eta(t)$ (the intensity of the η emission as a function of time).

$$\frac{1}{N}\frac{\Delta N_\eta}{\Delta t} \approx R_\eta(t) = R_\eta(0)\, e^{-t/\tau} = R_\eta(0)\, e^{-Rt}; \qquad R = \sum_\eta R_\eta(0). \tag{5}$$

$\Delta N_\eta(t_i)$ is the number of the decay products η registered by the η-detector during the time interval Δt around t_i.

In the theory the decay rates $R_\eta(t)$ are probabilities per unit time and the probabilities $P_\eta(t)$ are theoretically given by the Born probabilities of the observable Λ_η in the decaying state $\phi^D(t)$,

$$P_\eta(t) = \mathrm{Tr}(\Lambda_\eta |\phi^D(t)\rangle\langle\phi^D(t)|) = |\langle\psi_\eta|\phi^D(t)\rangle|^2 \quad \text{for} \quad \Lambda_\eta = |\psi_\eta\rangle\langle\psi_\eta|. \tag{6}$$

The partial decay rates (also called partial widths when multiplied by $\hbar$, $\Gamma_\eta = \hbar R_\eta(0)$) are (theoretically) the time derivatives of the probabilities $P_\eta(t)$

$$R_\eta(t) = \frac{d}{dt} P_\eta(t). \tag{7}$$

The experimental definition (5) of the lifetime τ, or of the total initial decay rate $R(0) = R = \frac{1}{\tau}$ using (5), is based on the validity of the exponential law for the decay probability $P_\eta(t)$. Though the exponential decay law (5) is time-honored, one often talks of deviations from the exponential decay law.

The reason for this concern with deviations from the exponential is a mathematical theorem (Kalfin, 1957, 1958) which states that there is no vector in Hilbert space that has an exact exponential time evolution. That means if one wants the exponential law of (5) to hold for $R_\eta(t)$ given by (7) one cannot take for $\phi^D(t)$ a Hilbert space vector. One has to take a vector $\phi^D \to \psi^{\mathrm{G}}$ with the properties

$$H\psi^{\mathrm{G}} = z\psi^{\mathrm{G}}, \qquad z = \left(E - i\frac{1}{2\tau}\right) \tag{8}$$

$$\psi^{\mathrm{G}}(t) = e^{\frac{-iHt}{\hbar}}\psi^{\mathrm{G}}(0) = e^{\frac{-izt}{\hbar}}\psi^{\mathrm{G}}(0). \tag{9}$$

This vector ψ^{G}, which needs to be properly defined, is called a Gamow vector.

The phenomenology of quantum physical decay, in particular the identity of the total initial decay rate $R = \sum R_\eta$ with the inverse lifetime $\frac{1}{\tau}$, $R = \frac{1}{\tau}$, depends upon the validity of the exponential law. The identity $\sum R_\eta = \frac{1}{\tau}$ would not hold if there were deviations from the exponential law.

For the initial decay rates $R_\eta(0)$ in (5) the Dirac Golden Rule (Fermi, 1950) had been used with great success. It relates $R_\eta(0)$ approximately to the matrix element $|\langle E_\eta|V|\phi^D\rangle|^2$ of the interaction Hamiltonian $V = H_n - H_0$. Using for ϕ^D the Gamow vector ψ^G of (9) one can derive in an heuristic manner, using the Lippmann–Schwinger equation, an exact Golden Rule (Bohm, 1979, 1994) which gives for the decay rate (7) the exponential time dependence of (5). Thus for the exponential law which is the basis of concepts like lifetime and partial rate, a Gamow vector (8) and (9) needs to be used, not a Hilbert space vector ϕ^D.

Another question is the relation of $R \equiv \frac{1}{\tau}$ to the width Γ which appears in the Breit–Wigner of (4).

The inverse lifetime $1/\tau$ of the exponential decay rates (5) and the width Γ of the Breit–Wigner energy distribution of (4) are conceptually and experimentally different quantities. Resonances of a scattering process are characterized by the resonance energy E_R (or resonance mass M in the relativistic case) and the resonance width Γ. The width Γ can be determined experimentally from (4) when Γ/E_R (and likewise Γ/M) are of the order $\sim 10^{-1} \ldots 10^{-4}$, and one often quotes the calculated quantity $\hbar/\Gamma = \tau^{\text{calc}}$ as the lifetime of the resonance. Decaying states of the decay process are characterized by the lifetime τ and the energy E_D. The lifetime $\tau = 1/R$ can be determined experimentally from (5) for $\hbar R/E_R \leq 10^{-10}$, and one often calls the calculated quantity $\Gamma^{\text{calc}} = \hbar R = \hbar/\tau$ with τ measured by (5), the width of the decaying particle, and $\hbar R_\eta \equiv \Gamma_\tau$ is usually called the partial width.

The question therefore arises, are resonances and decaying particles different physical entities or are they only quantitatively different in the magnitudes of Γ/E and $\hbar R/E$ with resonance width Γ and decay rate R just being different appearances of the same physical entity? In other words is $\Gamma = \hbar R$? Or is a resonance conceptually different from a decaying particle?

Many people think that resonances and decaying states are the same; especially in nonrelativistic quantum physics.

A contrary opinion predominates for the relativistic case: Resonances are complicated objects and cannot be described simply as states characterized by two numbers (M, Γ). They have a more complicated lineshape or at least an energy-dependent width $\Gamma(E_{\text{CM}})$ with *the* width defined as $\Gamma = \Gamma(E_{\text{CM}} = M)$. A common assumption is that one has

$$\frac{\hbar}{\Gamma} \approx \tau \equiv \frac{1}{R}. \tag{10}$$

This relation is based on the Weisskopf–Wigner approximation (Weisskopf and Wigner, 1930) of which M. Levy wrote in 1959 "There does not exist . . . a rigorous theory to which these various methods [Weisskopf–Wigner] can be considered as approximations" (Levy, 1959). Still the belief in the equality of the resonance width Γ and the decay rate R has been so thoroughly accepted that the terms width and rate are used interchangeably. A rigorous theory is needed to obtain the lifetime-width relation (10) as an exact result. We shall show that this can be done by separating the resonance part in an appropriate way from the nonresonant background not only in the nonrelativistic case but also in a relativistic theory of resonances and decay.

3. TIME ASYMMETRY FROM SIMPLE MATHEMATICS

Since Γ is measured by the Breit–Wigner cross-section of (4) and $1/\tau$ (equal to R if the exponential law holds) is measured by the exponential (partial) decay rates (5), the theory that one has to find must relate a Breit–Wigner amplitude $a_j^{\text{BW}}(E)$ with a Gamow vector ψ^{G}. Before this can be done, both these quantities need to be mathematically defined, since (1) ψ^{G} with properties (8) and (9) cannot be a Hilbert space vector and (2). the phenomenological Breit–Wigner (4b) for which the energy extends over the "physical" values $0 \leq E < \infty$ cannot be related to an exponential are (Erdely, *et al.*, 1954)

$$\begin{aligned} \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega t}}{\omega - (E_R - i\Gamma_R/2)} &= \theta(t)\, e^{i(E_R - i\Gamma_R/2)t} \\ &= \begin{cases} e^{-iE_R t} e^{-\Gamma E_R t/2} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \end{aligned} \tag{11a}$$

and

$$\frac{i}{\omega - (E_R - i\Gamma_R/2)} = \int_0^{\infty} dt\, e^{-iE_R t} e^{-\Gamma E_R t/2} e^{i\omega t}, \quad -\infty < \omega < \infty. \tag{11b}$$

This means that mathematics requires that the energy ω range over the entire real axis $-\infty < \omega < +\infty$ and the values of time range only from $0 < t < +\infty$. In contrast, in the traditional quantum theory in Hilbert space the time evolves over $-\infty < t < +\infty$ and energy (spectrum of the Hamiltonian H) is bounded from below $0 < \omega < +\infty$. The usual textbook derivation of the exponential time evolution for vectors with the Breit–Wigner energy distributions is one example of the "approximate" character of quantum mechanical derivations for scattering and decay phenomena.

The mathematical result (11) shows us which way we have to go to obtain a mathematical theory of quantum mechanical scattering and decay: the energy must be continued from the "physical" values $0 \leq E < \infty$ of (4b) into the complex energy plane, in particular to the negative values of (11b) and the exponential time evolution (11a) is asymmetric, it starts at a "beginning" $t = t_0 = 0$. Thus using the simple but exact mathematical relation (11), not the usual approximations of textbooks, shows already what many have felt about decay processes (Antoniou, 1992; Antoniou and prigorine, 1993; Cohen-Tannoudji *et al.*, 1997; Haag, 1990, 1997; Lee, 1981; Merzabcher, 1970; van Kampen, 2002), namely that the time evolution is asymmetric $t > t_0$ and the decay is an irreversible process.

4. DISTINCT SPACES FOR STATES AND OBSERVABLES—TAQT

To define the Gamow vector ψ^{G} as energy eigenkets of a self-adjoint Hamiltonian H, but with complex generalized eigenvalue $z = E - i\frac{1}{2\tau}$ one cannot use the RHS (3) of Schwartz type for which the energy wave functions in (2) are the Schwartz space functions $\phi(E) = \langle E|\phi\rangle \in \mathcal{S}$. Not all Schwartz space function can be analytically continued to complex energy plane. Since we want complex energy values, we need kets and wave functions that can be analytically continued into the complex energy plane. As a guide to find the space of analytic wave functions we use the well-known empirical concepts of quantum scattering. The Lippmann-Schwinger kets $|E^+\rangle$ and $|E^-\rangle$ are the eigenvectors of the (self-adjoint) Hamiltonian H which fulfill the boundary conditions given by the Lippmann-Schwinger (integral) equation (Newton, 1982). They have already energy values with "infitesimal imaginary part" $|E^{\mp}\rangle = |E_{\mp i\epsilon}\rangle$, $\epsilon > 0$, this means the complex conjugate of the energy wave functions $\langle f|E^{\mp}\rangle = \langle f|E_{\mp i\epsilon}\rangle \equiv \overline{f^{\mp}}(E)$ can be continued into the lower (for $^-$) and upper (for $^+$) complex energy plane, or $f^{\mp}(E) = \overline{\overline{f^{\mp}(E)}} = \langle^{\mp}E|f\rangle$ can be continued into the upper (for $^-$) and lower (for $^+$) energy plane. From this we conjecture that the energy wave functions for the outgoing particles $\langle^-E|\psi^-\rangle$ and the energy wave functions of the prepared in-states $\langle^+E|\psi^+\rangle$ form two distinct spaces of Schwartz functions which can be analytically continued into the upper and lower complex energy plane, respectively. To make this into a precise hypothesis, we postulate

The set of prepared in-state wave functions on the positive real semi-axis $E \in \mathbb{R}_+$ $\{\langle^+ E|\phi^+\rangle\} = \mathcal{S} \cap \mathcal{H}^2_-|_{\mathbb{R}_+}$ are smooth Hardy functions[2] of the lower energy plane. (12_-)

The set of observed (detector defined) out-particles wave functions $\{\langle^- E|\psi^-\rangle\} = \mathcal{S} \cap \mathcal{H}^2_+|_{\mathbb{R}_+}$ are smooth Hardy functions of the upper complex energy plane. (12_+)

One can show that two spaces of Hardy functions $\mathcal{S} \cap \mathcal{H}^2_+|_{\mathbb{R}_+}$ and $\mathcal{S} \cap \mathcal{H}^2_-|_{\mathbb{R}_+}$ (on the positive real energy axis $\mathbb{R}_+$) form together with $L^2(\mathbb{R}_+)$ a pair of RHSs, the Gadella RHSs (Gadella, 1983)

$$\mathcal{S} \cap \mathcal{H}^2_\mp|_{\mathbb{R}_+} \subset L^2(\mathbb{R}_+) \subset (\mathcal{S} \cap \mathcal{H}^2_\mp|_{\mathbb{R}_+})^\times. \tag{$13_\mp$}$$

This means that the set of outparticle "state" vectors ψ^- is given by the basis vector expansion

$$\psi^- = \sum_{j,n} \int_0^\infty dE|E, j, \eta^-\rangle\langle E, j, \eta|\psi^-\rangle \tag{14}$$

using the set of energy wave function $\langle^- E, j, \eta|\psi^-\rangle \equiv \langle^- E|\psi^-\rangle$ that fulfills (12_+). The set of these vectors ψ^- form an abstract linear space which we call Φ_+. This is the abstract Hardy space of vectors $\psi_1^-, \psi_2^-, \ldots$ which is mathematically (algebraically and topologically (has the same meaning of convergence)) equivalent to the function space (12_+). $\mathcal{S} \cap \mathcal{H}^2_+|_{\mathbb{R}_+}$ is called a realization of the abstract space Φ_+ in the same way as $L^2(\mathbb{R}_+)$ is the realization of the abstract Hilbert space $\mathcal{H}$ by the space of Lebesgue square integrable functions.

Equivalently, the set of the vectors ϕ^+ given by

$$\phi^+ = \sum_{j,\eta} \int_0^\infty dE|E, j, \eta^+\rangle\langle^+ E, j, \eta|\phi^+\rangle \tag{15}$$

using the set of energy wave functions $\langle^+ E, j, \eta|\phi^+\rangle \in \mathcal{S} \cap \mathcal{H}^2_-|\mathbb{R}_+$ form an abstract linear topological space which we call Φ_-. Therewith one has a pair of RHSs of Hardy type

$$\Phi_\mp \subset \mathcal{H} \subset \Phi_\pm^\times. \tag{$16_\pm$}$$

The same Hilbert space is equipped with two different Hardy spaces and their duals $\Phi_\pm^\times$ (space of antilinear continuous fuctionals). The Lippmann-Schwinger kets are then mathematically defined as functionals.

If one has only an RHS of Schwartz type (3) then one has just one kind of vectors connected with the experimental apparatus, the Schwartz space Φ. This gives us no possibility to distinguish between the set of vectors ϕ, which represent a state $|\phi\rangle\langle\phi|$ prepared by a preparation apparatus like the prepared in-state of a (resonance) scattering experiment and the set of vectors ψ which represent an

[2] Hardy functions are analytic, sufficiently decreasing functions on the semiplane (Bohm *et al.*, 1997, Appendix).

observable $|\psi\rangle\langle\psi|$ which is registered by a detector like the out-particles (decay products) of a resonance scattering experiment. The axiom of standard quantum mechanics therefore states that the set of prepared in-states and the set of the detected observables (or "out-states") are both the same (asymptotic completeness) and represented mathematically by the Schwartz space Φ:

$$\{\text{set of prepared in-states } \phi\} = \{\text{set of registered outobservables } \psi\} = \Phi \subset \mathcal{H}. \tag{17}$$

In its orthodox von Neumann form, this axiom even says that

$$\{\phi\} = \{\psi\} = \mathcal{H}. \tag{17$_1$}$$

Since the out-states, e.g., the decay products, are specified by the detector that registers them, they are really not states but observables. The new hypothesis that we conjectured from the heuristic meaning of the Lippmann-Schwinger kets distinguishes mathematically between states and observables. It postulates that the set of prepared states, defined by the preparation apparatus (accelerator), is described by

$$\{\phi^+\} = \Phi_- \subset \mathcal{H} \subset \Phi_-^\times \tag{18}$$

and the set of registered observables, defined by the registration apparatus (detector), is described by

$$\{\psi^-\} = \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times, \tag{19}$$

where $\mathcal{H}$ in (18) and (19) denotes the same Hilbert space but Φ_- and Φ_+ are distinct Hardy spaces which are dense in the same $\mathcal{H}$ (see (A-14) and (A-15)).

The Lippmann-Schwinger kets have now a precise mathematical meaning, namely the solutions of the generalized (see (A-7)) eigenvalue equation

$$H|E, j, n^{\mp}\rangle = E|E, j, \eta^{\pm}\rangle, \qquad 0 \geq E < \infty;$$
$$\text{with the boundary condition} \quad |E, j, n^{\pm}\rangle \in \Phi_\pm^\times. \tag{20}$$

The spaces $\Phi_\pm^\times$ contain many more elements than the Lippmann-Schwinger kets with real eigenvalue E. In particular $\Phi_+^\times$ contains the eigenkets

$$H|z, \ldots^-\rangle = z|z, \ldots^-\rangle \quad \text{for many} \quad z \in \mathbb{C}_-, \tag{21}$$

where $\mathbb{C}_-$ denotes the lower complex semiplane (because $\langle^- z|\psi^-\rangle$ is by (12$_+$) Hardy in the upper complex semiplane $\mathbb{C}_+$ and thus $\langle\psi^-|z^-\rangle = \overline{\langle^- z|\psi^-\rangle}$ is Hardy in the lower complex semiplane $\mathbb{C}_-$). For the complex semiplane in the definition of the Hardy spaces (12$_\mp$) one takes the second (or higher) sheet of the Riemann surface for the $\mathcal{S}$-matrix, the sheet on which the resonance poles are located and the z in (21) are the nonsingular points.

The heuristic description of quantum scattering using Lippmann-Schwinger kets therefore suggests that the axiom (17) (and in particular the Hilbert space axiom (17_1)) be replaced by a new axiom given by the hypotheses (18) and (19). The new axiom distinguishes also *mathematically* between prepared states defined by a preparation apparatus (accelerator) and the registered observables defined by the detector. This means that the set of in-states $\{\phi^+\}$ and the set of outobservables $\{\psi^+\}$ of a scattering experiment are *different* (dense) subspaces of the same Hilbert space $\mathcal{H}$.

The hypotheses (18) and (19) are the only new basic assumption of TAQT, all other axioms remain the same. In particular the dynamical equations are the same as before. In the Schrödinger picture the observables are kept time-independent and the state vector $\phi^+(t)$ obeys the Schrödinger equation

$$i\hbar\frac{\partial\phi^+(t)}{\partial t} = H\phi^+(t), \quad \phi^+(t = t_0 = 0) = \phi_0^+ \in \Phi_-. \tag{22}$$

In the Heisenberg picture the state is kept time-independent and the observable $\Lambda(t)$, or in the special case $\Lambda = |\psi^-\rangle\langle\psi^-|$, the observable vector $\psi^-(t)$ obeys the Heisenberg equations

$$i\hbar\frac{\partial\psi^-(t)}{\partial t} = -H\psi^-(t), \quad \psi^-(t = t_0 = 0) = \psi_0^- \in \Phi_+. \tag{23}$$

The difference with the traditional theory comes from the solutions of the dynamical equation due to the different boundary conditions. With the new Hardy space boundary conditions (18) the solutions for the states $\phi^+ \in \Phi_-$ are given by

$$\phi^+(t) = e^{-iHt}\phi^+ \equiv U_-^\dagger(t)\phi^+; \qquad 0 \le t < \infty. \tag{24}$$

And with the boundary condition (19) for the observables $\psi^- \in \Phi_+$ the solutions of (23) are given by[3]

$$\psi^-(t) = e^{iHt}\psi^- \equiv U_+(t)\psi^-; \qquad 0 \le t < \infty. \tag{25}$$

Thus in place of the unitary group solution with $-\infty < t < +\infty$, which one obtains from the same dynamical Eqs. (22) and (23) with the Hilbert space boundary condition $\psi \in \mathcal{H}, \phi \in \mathcal{H}$ (Stone–von Neumann theorem) one obtains under the new Hardy space boundary conditions (18) and (19) the semigroup solution (24) and (25).

[3] Usually one reserves the word "observables" for operators like $|\psi_i^-\rangle\langle\psi_i^-\rangle$ and mixtures thereof $\Sigma w^{ij}|\psi_i^-\rangle\langle\psi_j^-\rangle$; here we will also call the vectors $\psi_i^- \in \Phi_+$ observables. Precisely, the semigroup generator $H = H_+$ in (25) is the restriction of the self-adjoint operator H to the (dense in $\mathcal{H}$) subspace Φ_+ and the generator $H = H_-$ in (24) is the restriction of H to Φ_-. The same notation is used for $U(t)$. We often omit the subscripts of the operators which are the same as the subscripts of the spaces.

This is a radically new result. Its implications, the time ordering $t \geq t_0$, have been intuitively forseen by those not enamored with the Hilbert space axiom[4] (Antoniou, 1992; Antoniou and Prigogine, 1993; Cohen-Tannoudji *et al.*, 1977; Feynman, 1948, Gell-Mann and Hartle, 1994, 1995; Haag, 1990, 1997; Lee, 1981; Merzbacher, 1970; van Kampen, 2002). The Born probability for measuring the observable ψ^- in the state ϕ^+,

$$P_{\psi^-}(\phi^+(t)) = |(\psi^-|\phi^+(t))|^2 = |(\psi^-(t)|\phi^+)|^2 \tag{26}$$

are predicted for $t \geq t_0 = 0$ only. In the calculation (26) one can use either the Schrödinger picture (24) or the Heisenberg picture (25).

With the interpretation (Feynman, 1948; Gell-Mann and Hartle, 1994, 1995) that t_0 is the time at which the state ϕ^+ has been created, the time asymmetry (26) says that the observable $|\psi^-(t)\rangle\langle\psi^-(t)|$ can be detected in the state ϕ^+ only at times t after the state has been prepared. Thus the time asymmetry inherent in the Hardy space axiom (18) and (19) is an expression of causality for the Born probabilities.

With this interpretation, the asymmetric time evolution (24) and (25) that follows from the Hardy space axiom (mathematically based on the Paley–Wiener theorem for Hardy functions) appears quite acceptable and even welcome (Bohm, 1999; Gell-Mann and Hartle, 1994, 1995).

Historically, it was not the consideration of causality, but the desire to derive the exponential law for resonances, that led to the Hardy spaces (H. Baumgartel, personal communications, 1977) (Baumgartel, 1976; Bohm, 1978, 1981; Duren, 1970). This will be discussed next.

5. UNIFYING RESONANCES AND DECAYING STATES

The Born probability amplitude $(\psi_-|\phi^+)$ to register the observable $\Lambda^- = |\psi^-\rangle\langle\psi^-|$, $\psi^- \in \Phi_+$, in the state $\phi^+ \in \Phi_-$ is expressed using the standard notions of scattering theory as the matrix element of the S-operator:

$$(\psi^-|\phi^+) = (\Omega^-\psi^{\text{out}}|\Omega^+\phi^{\text{in}}) = (\psi^{\text{out}}|S\phi^{\text{in}}) = (\psi^{\text{out}}|\phi^{\text{out}}). \tag{27}$$

This is essentially the statement of standard scattering theory (Bohm, 1979, Newton, 1982; Weinberg, 1995) except that there one speaks of out-states ϕ^- instead of outobservables $\psi^- \equiv \Omega^-\psi^{\text{out}}$. But Born probabilities correlate observables and states, not states and other states, and the detector in scattering

[4] The possibility of two distinct spaces for the prepared states $\{\phi^+\}$ and for the registered observables $\{\psi^-\}$ is already contained in the historical paper of Feynman (1948). He distinguishes between the state at times $t' < t_0$ which is defined by the preparation (our prepared states ϕ^+) and what he calls "state characteristic of the experiment" at time $t'' > t_0$ (our registered observables ψ^-). He mentions the possibility $\{\psi^-\} \neq \{\phi^+\}$ in Footnote 14, attributing it to H. Snyder, but does not consider it. We implement this possibility by the choice of the two Hardy spaces Φ_- and Φ_+.

experiments is not built to the specifications of prepared states, but to the specification of the particles to be registered in the outregion, which are therefore observables. The matrix element $(\psi_{-}|\phi^{+})$ can now be expressed using (14) and (15) and with the use of symmetries (angular momentum, energy conservation) one obtains

$$(\psi^{-}|\phi^{+}) = \sum_{\eta,\eta',j} \int_{0}^{\infty} dE \langle \psi^{-}|E, j, \eta^{-}\rangle S_{j}^{\eta\eta'}(E) \langle^{+}E, j, \eta'|\phi^{+}\rangle, \tag{28}$$

where $S_{j}^{\eta} \equiv \langle^{-}E, j, \eta|E, j, \eta_{0}^{+}\rangle$ is the S-matrix element which is related to the scattering amplitude of (4) by

$$S_{j}^{\eta_0} = 2ia_{j}^{\eta_0}(E) + 1 \quad \text{(elastic channel from } \eta_0 \to \eta_0\text{)}, \tag{29a}$$

$$S_{j}^{\eta} = 2ia_{j}^{\eta}(E) \quad \text{(reaction channel from } \eta_0 \to \eta\text{)}. \tag{29b}$$

Under the new hypothesis $(12_{\mp})$ the energy wave functions are not only smooth square integrable functions but also analytic in such a way that the integral in the S-matrix element (28) can be continued into the lower half plane of the second sheet. The contour integration can therefore be deformed from the continuous spectrum of H ($0 \leq E < \infty$, the scattering energies) into a contour around the resonance pole and some background integral that corresponds to B in (4) (Bohm *et al.*, 1997).

In the integrals along the circles around each resonance pole at $z_{R_i} = E_{R_i} - i\Gamma_i/2$ of (28) one uses the expansion

$$S_{j}^{\eta\eta'} = \frac{R^{(i)}}{z - z_{R_i}} + R_0 + R_1(z - z_{R_i}) + \cdots$$

and obtains for the i-th pole term of (ψ^{-}, ϕ^{+}):

$$\begin{aligned}
(\psi^{-}|\phi^{+})_{\text{pole}_i} &\equiv \oint_{\hookleftarrow C_i} dz \langle \psi^{-}|z^{-}\rangle S(z) \langle^{+}z|\phi^{+}\rangle \\
&= \oint_{\hookleftarrow C_i} dz \langle \psi^{-}|z^{-}\rangle \frac{R^{(i)}}{z - z_{R_i}} \langle^{+}z|\phi^{+}\rangle \\
&= -2\pi i R^{(i)} \langle \psi^{-}|z_{R_i}^{-}\rangle \langle^{+}z_{R_i}|\phi^{+}\rangle \qquad (30a) \\
&= \int_{\infty II}^{\infty} dE \langle \psi^{-}|E^{-}\rangle \langle^{+}E|\phi^{+}\rangle \frac{R^{(i)}}{E - z_{R_i}}. \qquad (30b)
\end{aligned}$$

In this derivation one has made use of the Hardy property of the wave functions $\langle \psi^{-}|z^{-}\rangle$, $\langle^{+}z|\phi^{+}\rangle$ and used the Cauchy theorem for (30a) and the Titschmarsh theorem for (30b). Comparing (30a) with (30b) this leads to the following definition of the Gamow vectors $|z_{R_i}^-\rangle$ as functionals over all $\psi^- \in \Phi_+$:

$$\langle \psi^- | z_{R_i}^- \rangle = \frac{i}{2\pi} \oint_{\hookleftarrow C_i} dz \frac{\langle \psi^- | z^- \rangle}{z - z_{R_i}} = \frac{i}{2\pi} \int_{-\infty II}^{+\infty} dE \frac{\langle \psi^- | E^- \rangle}{E - z_{R_i}}. \tag{31}$$

The integral along the energy axis extends from $-\infty$ in the second sheet along the upper rim of the second sheet to $+\infty$ and the values along the cut $0 \leq E < \infty$ are the physical scattering energies. Since the values of $\langle \psi^- | E^- \rangle$ for negative E of Hardy functions are already determined by their values for $0 \leq E < \infty$[5] the values in (31) are determined from the scattering energies.

This means we have the following result (as a consequence of the Hardy space axiom): if we replace the phenomenological Breit–Wigner in (4b) which is measured only for $0 \leq E < \infty$ by the "exact" Breit–Wigner of (11) for which the energy extend from $-\infty_{II} < E < \infty$, then one can associate to it an ideal Gamow vector ψ_j^{G}, defined as the continuous superposition of the Lippmann-Schwinger kets $|E, j, \ldots^-\rangle$ with the "exact" Breit–Wigner as the wave function[6] of $\langle^- E, j \ldots | \psi_j^{\mathrm{G}} \rangle$

$$a_j^{\mathrm{BW}}(E) = \frac{r_\eta}{E - \left(E_R - i\frac{\Gamma}{2}\right)} \iff \psi_j^{\mathrm{G}} = \sqrt{2\pi\Gamma}|z_R, j \ldots^-\rangle$$
$$= \frac{i\sqrt{2\pi\Gamma}}{2\pi} \int_{-\infty II}^{+\infty} dE \frac{|E, j \ldots^-\rangle}{E - z_R} \tag{32}$$
$$-\infty_{II} < E < +\infty, \qquad z_R = E_R - i\frac{\Gamma}{2}.$$

In here the integral of (14) is extended from $0 \leq E < +\infty$ to $-\infty < E +\infty$ as in (11a). This "mistake" has already been done by Fermi (1932) to obtain causality which otherwise would have been lost (Hegerfeldt, 1994). For the vector ψ_j^{G} defined in (32), *and only if the integral extends to* $-\infty_{II}$, one can derive (using the property of Hardy functions) that

$$\langle H\psi_\eta^- | \psi^{\mathrm{G}} \rangle \equiv \langle \psi_\eta^- | H^\times | \psi^{\mathrm{G}} \rangle = \left(E_R - i\frac{\Gamma}{2}\right) \langle \psi_\eta^- | \psi^{\mathrm{G}} \rangle \quad \text{for all} \quad \psi_\eta^- \in \Phi_+, \tag{33a}$$

when $H = H_0 + V$ is self-adjoint (and semibounded). This justifies the notation $\psi_j^{\mathrm{G}} = |E_R - i\Gamma/2, j, \ldots^-\rangle$. In Dirac notation the arbitrary $\psi_\eta^- \in \Phi_+$ is omitted and (33a) is written as

$$H^\times \left| E_R - i\frac{\Gamma}{2}, j, \ldots^- \right\rangle = \left(E_R - i\frac{\Gamma}{2}\right) \left| E_R - i\frac{\Gamma}{2}, j, \ldots^- \right\rangle \tag{33b}$$

[5] see Appendix of Bohm (1997), van Winter theorem.
[6] The normalization factor $\sqrt{2\pi\Gamma}$ is an inconsequential convention.

Dirac also omitted the $^{\times}$ of the conjugate operator $H^{\times}$ which is uniquely defined as the extension of the operator $H^{\dagger} = H$ to $\Phi_{+}^{\times}$ by the first equality in (33a), cf. Appendix. The Gamow ket $|E_R - i\Gamma/2^{-}\rangle$ is thus something like a Dirac ket, but even more so since it can have complex eigenvalue z and z_R. For Gamow vector defined in (32) by the pole position, z_R is the position of the S-matrix pole of the resonance in the lower half plane. Unlike the usual Dirac ket, which is mathematically defined as a functional over the Schwartz space Φ, the Gamow ket (32) is an element of $\Phi_{+}^{\times} \supset \Phi^{\times}$. There are many more eigenkets (21) with complex eigenvalues z in $\Phi_{+}^{\times}$; Gamow vectors are the eigenkets *associated to the values* $z_{R_1}, z_{R_2}, \ldots$ of (first-order) *S-matrix poles.*

The Gamow vector with exact Breit–Wigner distribution defined by (32) represents the state associated to the Breit–Wigner scattering amplitude without the background B(E) of (4b); it thus represents the resonance pole. For this state vector one derives (Bohm, 1978, 1981; Bohm *et al.*, 1997) the exponential time evolution:

$$\psi^{\mathrm{G}}(t) \equiv e^{-iH^{\times}t}\psi^{\mathrm{G}} = e^{-iE_R t}e^{-\frac{\Gamma}{2}t}\psi^{\mathrm{G}}; \qquad \text{for } t \geq 0 \text{ only.} \tag{34}$$

Formally (34) is just (33) applied in the exponent. But to prove (34) as a functional equation, one has to show that

$$\begin{aligned} \langle e^{iHt}\psi^{-}|z_R^{-}\rangle &= \frac{i}{2\pi}\int_{-\infty}^{+\infty} dE\,\frac{\langle e^{iHt}\psi^{-}|E^{-}\rangle}{E - z_R} \\ &= \frac{i}{2\pi}\int_{-\infty}^{+\infty} dE\,\frac{\langle\psi^{-}|E^{-}\rangle e^{-iEt}}{E - z_R} = e^{-iz_R t}\langle\psi^{-}|z_R^{-}\rangle \quad \text{for} \quad t \geq 0. \end{aligned} \tag{35}$$

The latter equality follows only if for $\langle\psi^{-}|E^{-}\rangle \in \mathcal{S} \cap \mathcal{H}_{-}^{2}$ also $\langle\psi^{-}|E^{-}\rangle e^{-iEt} \in \mathcal{S} \cap \mathcal{H}_{-}^{2}$, and this is only the case for $t \geq 0$. Thus for this derivation again the mathematical properties of the Hardy functions are needed (Bohm, 1978, 1981; Bohm *et al.*, 1997) and the time asymmetry $t \geq 0$ emerges as a consequence of the axiom (18) and (19).

The Γ in (34) is the same as the Γ in (32), that means it is the width of the Breit–Wigner used in fits of the crosssection (4) to the experimental cross-section data. The comparison of (34) and (33) with (8) (9) shows that Γ is also the inverse lifetime determined from the fit of the decay rates to the exponential (5), if one takes for the decaying state ϕ^{D} in (6) the Gamow vector ψ_j^{G} defined in (32). In contrast to the superposition (14) and (15) for ordinary vectors $\psi^{-} \in \Phi_{+}$, the integration in (32) extends over $-\infty_{II} < E < \infty$ and this is only possible if ψ^{G} is a functional $\psi^{\mathrm{G}}(\psi^{-}) = \langle\psi^{-}|\psi^{\mathrm{G}}\rangle$ over the Hardy space, $\psi^{-} \in \Phi_{+}$.

The ideal Gamow vector in (32) with the "exact" Breit–Wigner energy distribution is therefore the state vector of the resonance per se and (34) shows that it obeys an exact exponential decay law.

The scattering amplitude $a_j(E)$ need not just have *one* Breit–Wigner; in fact if there are N resonances at $z_{R_i} = E_{R_i} - i\Gamma_i/2, i = 1, \ldots, N,$ in the j-th partial wave, then one can show that the S-matrix element is given (also as a consequence of the Hardy space axiom) by

$$2ia_j^\eta(E) = S_j^\eta(E) = b(E) + \sum_i \frac{R_\eta^{(i)}}{E - z_{R_i}}, \tag{36}$$

where $b(E) = B(E)/(2i)$ is a background amplitude as in (4). Thus the scattering amplitude into channel η is a superposition of N Breit–Wigners. The interference term that follows from (36) has been observed in several experiments (nuclear physics (Nathan *et al.*, 1975; von Brentano, 1996)[7], the $\rho - \omega$ system).

If there are N resonances in the j-th partial wave, one also derives (from the Hardy space axiom) for the prepared state vector ϕ^+ an alternate basis vector expansion to the Dirac expansion (15). This is given by

$$\phi^+ = \phi^{\text{bg}} + \sum_i |z_{R_i}, j, \ldots^-\rangle c_{R_i}, \quad c_{R_i} = \frac{2\pi}{i} r^{(i)} \langle^+ z_{R_i}, j, \ldots |\phi^+\rangle, \tag{37}$$

where ϕ^{bg} is the functional on $\Phi_+^\times = \{\psi^-\}$ given by

$$\begin{aligned}\langle\psi^-|\phi^{\text{bg}}\rangle &= \int_0^{-\infty II} dE \langle\psi^-|E^-\rangle\langle^+E|\phi^+\rangle S_{II}(E) \\ &= \int_0^{+\infty} dE \langle\psi^-|E^-\rangle\langle^+E|\phi^+\rangle b(E).\end{aligned} \tag{38}$$

The last equality follows from the van Winter theorem and $b(E)$ is a (slowly varying) function of E uniquely determined by $S_{II}(E)$ on the positive real axis (Gadella, 1997). The vectors of (32) $|z_{R_i}, j, \eta^-\rangle$ are the Gamow vectors representing the resonances at z_{R_i}. The "complex basis vector expansion" (37) expresses the prepared state ϕ^+ as a superposition of the Gamow vectors plus some vector ϕ^{bg} given by (38). Here $b(E) = 2iB(E)$ ($b(E) - 1 = 2iB(E)$ for elastic channel $\eta = \eta_0$) is the phenomenological amplitude $B(E)$ of (4).

This shows that to each Breit–Wigner in the scattering amplitude (36) there corresponds an exponentially evolving Gamow ket in (37) and if there is more than one Gamow ket in (37) (e.g., in the neutral Kaon system) one has an interference term in the decay rate. But, depending upon the preparation of the state ϕ^+, one can also have a background vector ϕ^{bg} (related to the background $B(E)$ in (4)). This vector ϕ^{bg} does not evolve exponentially in time.

The Weisskopf-Wigner approximation (omitting the energy continuum) results in a superposition of a finite number of Breit–Wigner in (36) and a superposition of a finite number of corresponding Gamow vectors in (37), each term in

[7] This paper uses finite complex effective Hamiltonian omitting the background ϕ^{bg} and/or $b_j(s)$.

(37) having a corresponding term in (36). There is ample experimental evidence for the interference terms following from these superpositions (Baldo *et al.*, 1957; Bollini *et al.*, 1996; Ferreira, 1989; Kukulin *et al.*, 1989; Lee *et al.*, 1957; Nathan, 1975; Tolstikhin *et al.*, 1998; Vertese *et al.*, 1989; von Brentano, 1996)[8].

6. DISCUSSION AND CONCLUSIONS

In traditional quantum theory there is no possibility for a precise mathematical representation of *resonance* and/or *decaying* state. Some physicists—especially in relativistic particle physics—therefore doubt the usefulness of considering a quasistable particle as autonomous entity, as one does for stable states. A resonance is considered as a complicated phenomenon that cannot be idealized as a single physical object.

The practice is, however, quite different. Resonances are classified in the same way as stable states, except that in place of the energy E_n, or mass M for relativistic particles, one uses two numbers (E_R, Γ) or (M, Γ), where Γ is determined experimentally by a fit to the line shape (4). Decaying particles are also characterized by two numbers (E_R, τ) or (M, τ) where τ is determined experimentally by a fit to the exponential decay rate (5). Stable particles are quantum states for which $\Gamma = 0$ or $\tau = \infty$ and decaying particles are resonances with long lifetime $\approx \hbar/\Gamma$.

The theoretical methods by which one establishes the connection between lifetime τ and width Γ, (the Weisskopf-Wigner methods) are approximate. By some complicated calculations one can derive (Goldberger and Watson, 1964) for the decay probability of a resonance (E, Γ) an exponential decay law with a lifetime $= \hbar/\Gamma$; but this involves various assumptions, uses undefined mathematics and the decay probability is only nearly equal to the exponential because, a "second" term has to be omitted (Goldberger and Watson, 1964, chap. 8, Eq. (116)). The problem thus is to isolate the resonance term from a second term in the scattering amplitude in such a way that the resonance term describes the resonance per se; the second term then is the nonresonant background (if there are no other resonance poles or singularities in the partial wave). This is accomplished by the choice of the "exact" Breit–Wigner in (32). Similarly, from the state vector ϕ one isolates the Gamow vector (31) and attributes the deviations from the exponential law to the background ϕ^{bg}. In the same way as one makes the idealization of an isolated stable state, we make the idealization of a quasi-stable state, which in the scattering experiment is the resonance per se represented by the exact Breit–Wigner of (32). This is according to (28) and (30) the S-matrix definition of a resonance by a first-order pole at $z_R = E_R - i\Gamma/2$, it leads to the "ideal" Gamow vector of (32). This vector has a long tradition (Gamow, 1928;[9] Siegert, 1939; Thomson, 1884)

[8] See footnote 7.

[9] Complex eigenfunctions have been used earlier for transient modes of an electromagnetic filed.

and a mathematically bad reputation, worse than the Dirac ket. But it makes good sense a functional on the Hardy space.

The choice of the parameterization of the scattering amplitude is important for the definition of resonance mass and width. Choosing for the scattering amplitude (4a) with (4b) fixes (to a certain extent for the relativistic resonances; Kielanowski, 2003) the definition of the resonance energy E_R (or M) and the resonance width Γ. For example, for the π-N resonance Δ the fit to (4a) with (4b) leads to what is called the pole position mass $M = 1210(1)$ MeV and pole position width $\Gamma = 100(1)$ MeV (Particle Data Group, 2002). In addition the Particle Data Table (Particle Data Group, 2002) also gives what it calls the "Breit–Wigner mass and width" $M_\Delta = 1223(1)$ MeV, $\Gamma_\Delta = 120(1)$ MeV which is obtained if one fits the line shape data to the so-called "relativistic Breit–Wigner with energy-dependent width" of the on-mass-shell renormalization scheme (Kielanowski, 2003). The difference in these values is about $10 \times$ experimental error (1 MeV) and the right choice for the parameterization of $a_j(E)$ is significant.

The mathematical method within the traditional Hilbert space axioms of quantum mechanics that came closest to defining (8) are the spectral deformation techniques (Balsev, 1984; Reed and Simon, 1978). These methods have been used for many applications in atomic and molecular physics and in quantum chemistry (Brandas, 1986; Reinhardt, 1982; Simon, 1978, 1979). The resonances are described by square integrable eigenvectors $\psi_n(\theta)$, $H(\theta)\psi_n(\theta) = z_n\psi_n(\theta)$, of a non-self-adjoint Hamiltonian $H(\theta) = U(\theta)(H_0 + V)U^{-1}(\theta)$ with complex eignvalue z_n. Here the potential $U(\theta)VU^{-1}(\theta)$ is analytic for certain complex values of θ. Vectors with the property (8) and (9) can then be defined by

$$\psi^{\mathrm{G}} = U^{-1}(\theta)\psi_n(\theta). \tag{39}$$

For stable states one can define state vectors as eigenvectors of the time-independent Schrödinger Eq. (8) under very specific boundary conditions (e.g., the eigenstates ϕ_n of the harmonic oscillator Hamiltonian with the condition that $\phi \in$ Schwartz space). For resonance states this is not possible, the Hamiltonian operator and boundary conditions alone do not specify a resonance state. For instance many analytic continuations (21) of the Lippmann-Schwinger kets $|E^-\rangle$ into the complex semiplane $\mathbb{C}_-$, $|z^-\rangle \in \Phi_+^\times$ fulfill (8) (and (9)) but they do not represent resonance states.

In the complex scaling methods one uses a pair of operators, the Hamiltonian H and the decay-interaction operator V (or $H_0 = H - V$), and characterizes the resonance state by a solution of the Lippmann-Schwinger equation. We characterize the resonance state by the resonance part (Breit–Wigner) of the scattering amplitude or of the S-matrix.

One can also show that there is a connection between the RHS definition and the methods of spectral deformation: in the special case that H_0 has a discrete eigenvalue E_n^0 in the continuous spectrum $\{E^0|0 \leq \infty\}$ and $H = H_0 + \lambda V$ has the same continuous spectrum, the eigenvalue E_n^0 can change into a complex generalized eigenvalue z_n of H with the property that $z_n \to E_n^0$ for $\lambda \to 0$. For this special case (Friedrich's model) one can show (Antoniou *et al.*, 2001) that the Gamow vectors defined as generalized eigenvectors of H with complex eigenvalue z_n are the same as those given by (39).

To obtain a meaningful theory of resonance and decay phenomena, one needs a vector with the following properties:

1. It must have a Lorentzian or Breit–Wigner energy distribution.
2. It must have an exponential time evolution.
3. The parameter of the Lorentzian Γ and the parameter of the exponential τ must be related by $\tau = \hbar/\Gamma$.

The vectors which have these properties are the Gamow vectors of (32) defined by (31) as a functional over the Hardy space Φ_+. Such vectors cannot exist in Hilbert space. They cannot even exist as generalized eigenvector defined as functional over the Schwartz space, like the usual Dirac ket $|E\rangle \in \Phi^\times$ of (3).

The kets that one needs are suggested by the Lippmann-Schwinger equations. The Lippmann-Schwinger kets (of which there are two kinds $|E^\mp\rangle$, require some analyticity properties. We give them a mathematical meaning by defining them as functional over the Hardy spaces of $(16_\pm)$:

$$|E^\mp\rangle \in \Phi_\pm^\times.$$

They can be analytically continued to $|z^\mp\rangle \in \Phi_\pm^\times$ for $z \in \mathbb{C}_\mp^R$. For $\mathbb{C}_\mp^R$ one takes the nonsingular points of the lower or upper complex semiplane of the second sheet of the Riemann energy surface for the S-matrix $S_j^\eta(E)$.

The Gamow state vectors are associated with the singular point of the analytically continued Lippmann-Schwinger kets $|z^-\rangle$. The Gamow vector (32) represents a first-order resonance. But there could also be higher-order resonances represented by Gamow–Jordan vectors which would correspond to higher-order poles of the S-matrix, if they exist in nature (Antoniou *et al.*, 1998; Bohm *et al.*, 1997).

The Hardy space hypotheses (18) and (19), which replaces the Hilbert space axiom (17), are the only new assumption we make in addition to the traditional assumptions of quantum theory. The unified theory of resonances and decay requires this new axiom. The new complex eigenvalue resolution (37) and the corresponding expansion (36) of the scattering amplitude or S-matrix can only be obtained if one uses the Hardy space hypotheses (18) and (19). The Weisskopf-Wigner approximations of (37) and (36)—i.e., omitting the continuum (38) or setting $b(E) = 0$—lead to the effective theories with finite complex Hamiltonian matrices, which have been successfully applied in different areas of quantum physics (Baldo *et al.*, 1987; Bollini *et al.*, 1996; Ferreira, 1989; Kukulin *et al.*, 1989; Lee *et al.*, 1957; Nathan *et al.*, 1975; Vartse *et al.*, 1989; von Brentano, 1996). The Hardy space quantum mechanics is thus the theory of which the Weisskopf-Wigner methods are approximations, and (37) and (36) show what these approximations mean.

The axioms of nonrelativistic timeasymmetric quantum mechanics can be extended to a relativistic theory by an appropriate extension of the Hardy space axiom into the relativistic domain. There the hypotheses (18) and (19) will lead to new predictions. This relativistic theory of resonance scattering and decay, in which the time evolution semigroup is generalized to the causal Poincaré semigroup (Bohm, 2003), will be discussed in an other contribution to this volume (Kielanowski, 2003).

APPENDIX: RIGGED HILBERT SPACES

RHSs, also called Gelfand triplets, are triplets of linear spaces, which differ from each other by their topology. In other words the meaning of convergence is different for each space, which implies that the limit points of converging sequences are different in the three spaces that make up the Gelfand triplet. The three spaces have properties which a physicist would call a Hilbert space but only one of them is a Hilbert space by the mathematical definition, i.e., it is complete with respect to the Hilbert space convergence.

One starts with a linear scalar product space denoted by Φ_{alg} (also called a pre-Hilbert space). The subscript (alg) refers to the algebraic operations that one can perform in them, namely linear superpositions and the scalar product. The three spaces that form the RHS, denoted by

$$\Phi \subset \mathcal{H} \subset \Phi^{\times}, \tag{A1}$$

are obtained by completing the purely algebraic space Φ_{alg} with respect to three different topologies, i.e., three definitions of convergence. To obtain each space, one adjoins to Φ_{alg} the limit elements of Cauchy sequences, but one uses three different meanings of convergence and thus obtains three different complete spaces. The space with a stronger topology, i.e., a stronger definition of convergence is dense in the space with a weaker topology. The Hilbert space $\mathcal{H}$ is obtained by completing Φ_{alg} with respect to the norm, denoted by $\tau_{\mathcal{H}}$. The space Φ is obtained by completing Φ_{alg} with respect to a stronger topology than $\tau_{\mathcal{H}}$, denoted by τ_{Φ}. The third space, $\Phi^{\times}$, is the space of continuous antilinear functionals F on Φ,

$$|F\rangle : \phi \in \Phi \to F(\phi) = \langle\phi|F\rangle \in \mathbb{C}. \tag{A2}$$

Thus one obtains the triplet of spaces, or an RHS (A1).

In the Hilbert space, there is a one-to-one correspondence between elements of the space of antilinear functionals $\mathcal{H}^{\times}$ and elements of $\mathcal{H}$, thus one can identify them with each other:

$$\mathcal{H} = \mathcal{H}^{\times}. \tag{A3}$$

According to the Frechet–Riesz theorem, for every $|f\rangle \in \mathcal{H}^{\times}$ there is an $f \in \mathcal{H}$ such that $f(\phi) = \langle\phi|f\rangle = (\phi, f)$ for all $\phi \in \mathcal{H}$. The functional $\langle\phi|F\rangle$ is an extension of the scalar product (ϕ, f) to those $|F\rangle \in \Phi^{\times}$ which are not in $\mathcal{H}$.

Let A be a linear operator in Φ, continuous with respect to τ_{Φ}, and $A^{\dagger}$ its adjoint in $\mathcal{H}$. To the triplet of spaces (A1) corresponds a triplet of operators

$$A : A^{\dagger}|_{\Phi} \subset A^{\dagger} \subset A^{\times}, \tag{A4}$$

where $A^{\dagger}$ is the Hilbert space adjoint of A and $A^{\dagger}|_{\Phi}$ its restriction to Φ. If A is a continuous operator with respect to τ_{Φ}, it need not be, and in general is not a continuous (bounded) operator in $\mathcal{H}$. We shall only consider τ_{Φ}-continuous operators. So far the restriction to continuous operators in Φ has proven to be sufficient for quantum physics, whereas $\tau_{\mathcal{H}}$-continuous operators are not sufficient (e.g., the position and/or momentum operators cannot be an continuous operators in $\mathcal{H}$, neither can the generators of unitary representations of noncompact groups).

The conjugate operator, $A\times$, of the τ_{Φ}- *continuous* linear operator A is a continuous linear operator in $\Phi^{\times}$ defined by

$$\langle A\phi|F\rangle = \langle\phi|A^{\times}|F\rangle \qquad \forall\phi \in \Phi \quad \text{and} \quad \forall F \in \Phi^{\times}. \tag{A5}$$

It is a unique extension of the Hilbert Space adjoint operator

$$(A\phi, f) = (\phi, A^{\dagger}|f) \quad \text{for} \quad \phi, f \in \mathcal{H}. \tag{A6}$$

A vector $F \in \Phi^{\times}$ is called a generalized eigenvector of the τ_{Φ}- continuous operator A if for some $\omega \in \mathbb{C}$

$$\langle A\phi|F\rangle = \langle\phi|A^{\times}|F\rangle = \omega\langle\phi|F\rangle. \tag{A7}$$

This is also written as $A^{\times}|F\rangle = \omega|F\rangle$, or as Dirac did, $A|F\rangle = \omega|F\rangle$ for Hermitian A. An example of generalized eigenvectors are the Dirac kets. Their eigenvalues belong to the continuous spectrum of a self-adjoint H, $H^{\times}|E\rangle = E|E\rangle$, $0 \leq E < \infty$.

Kets (and all $F \in \Phi^{\times}$) depend on the choice for the space Φ. Dirac kets are usually defined with Φ as the Schwartz space S, i.e., the space of smooth, rapidly decreasing wave functions $\phi(E) = \langle E|\phi\rangle$.

The triplet of function spaces

$$S \subset L^2 \subset S^{\times}, \tag{A8}$$

where S is the space of smooth rapidly decreasing functions and L^2 is the space of Lebesgue square-integrable functions with scalar product given by the integral

$$(\psi, \phi) = \int_{-\infty}^{\infty} dE\,\overline{\psi(E)}\phi(E) = \int_{-\infty}^{\infty} dE\,\langle\psi|E\rangle\langle E|\phi\rangle, \tag{A9}$$

is an example of an RHS. It is called a realization of the abstract Schwartz–RHS

$$\Phi \subset \mathcal{H} \subset \Phi^{\times}, \tag{A10}$$

whose vectors $\psi, \phi \in \Phi$ are the vectors for which the Dirac basis vector expansion

$$\phi = \int_{-\infty}^{\infty} dE|E\rangle\langle E|\phi\rangle \tag{A11}$$

(omitting the arbitrary $\psi \in \Phi$ from (A9)) holds.

The integrals in (A9) can always be chosen as Riemann integrals if $\psi, \phi \in \Phi$, i.e., $\psi(E), \phi(E) \in S$ and we shall do so. The integral for the scalar product in L^2, however, must be a Lebesgue integral since the space of Riemann square integrable functions cannot be a complete Hilbert space.

The space Φ that together with $\mathcal{H}$ form an RHS cannot be an arbitrary topological space. The topology τ_Φ must fulfill certain additional conditions (e.g., nuclearity) in order that Dirac's basis vector expansion (A11) can be proven as the nuclear spectral theorem. These additional requirements on Φ are part of the definition of every RHS (A1). The Dirac basis vector expansion (A11) is the most important theorem for quantum mechanics; even before its proof, it had been used profusely in quantum theory. In this paper it appears in (14), (15), and (28).

Examples of other RHSs besides the Schwartz–RHS (A8) are the Hardy–RHSs. There are two Hardy–RHSs denoted

$$\Phi_+ \subset \mathcal{H} \subset \Phi_+^{\times}, \tag{A12}$$

$$\Phi_- \subset \mathcal{H} \subset \Phi_-^{\times}, \tag{A13}$$

and realized by the function spaces

$$\mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}_+} \subset L^2(\mathbb{R}_+) \subset (\mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}_+})^{\times}, \tag{A14}$$

$$\mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}_+} \subset L^2(\mathbb{R}_+) \subset (\mathcal{S} \cap \mathcal{H}_-^2|_{\mathbb{R}_+})^{\times}, \tag{A15}$$

respectively.

Here the Hilbert space, $L^2(\mathbb{R}_+)$, is the space of Lebesgue square integrable functions on the positive real line $\mathbb{R}_+$, and $\mathcal{H}_\pm^2 \cap S|_{\mathbb{R}_+}$ denotes the smooth rapidly decreasing functions $\psi^{\pm}(E)$, $E \in \mathbb{R}_+$, which can be analytically continued into the upper half (for $\mathcal{H}_+^2$) and the lower half (for $\mathcal{H}_-^2$) complex energy plane. More precisely, the $\psi^{\mp}(E)$, $E \in \mathcal{H}_\pm^2 \cap S|_{\mathbb{R}_+}$ are the boundary values of smooth analytic functions in the lower $(-)$ and upper $(+)$ complex half plane that decrease sufficiently fast at the infinite semicircle (for the definition, see [35] Appendix). We call the spaces $\Phi_\pm$ and their realization $\mathcal{H}_\pm^2 \cap S|_{\mathbb{R}_+}$ Hardy spaces. One can show that these function spaces (A14), (A15), also form an RHS [34]. The Hardy–RHSs are needed if one wants to consider generalized eigenvectors of the Hamiltonian

H belonging to the continuous spectrum

$$H^{\times}|E^{\pm}\rangle = E|E^{\pm}\rangle \quad 0 \leq E < \infty \tag{A16}$$

and fulfilling outgoing (−) and incoming (+) boundary conditions (e.g., the solutions of the Lippmann-Schwinger equations). For these generalized eigenvectors, we have $|E^{\pm}\rangle \in \Phi_{\mp}^{\times}$ but $|E^{\pm}\rangle$ are not elements of the dual of the Schwartz space $\Phi^{\times}$. There are many other examples of generalized vectors that are in $\Phi_{\pm}^{\times}$ and not in $\Phi^{\times}$. For example, the generalized eigenvectors of the self-adjoint Hamiltonian H with complex eigenvalue, the Gamow vectors

$$H^{\times}|E_R - i\Gamma/2^{-}\rangle = (E_R - i\Gamma/2)|E_R - i\Gamma/2^{-}\rangle, \tag{A17}$$

are elements of $\Phi_{\pm}^{\times}$, but not elements in $\Phi^{\times}$.

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REFERENCES

Antoniou, I. (1992). *Proceedings 2nd International Wigner Symposium, Goslar 1991,* H.-D. Doebner *et al.*, eds., World Scientific, Singapore.

Antoniou, I., Gadella, M., and Pronko, G. P. (1998). *Journal of Mathematical Physics* **39**, 2459.

Antoniou, I., Melnikov, Y., and Yarevski, E. (2001). *Chaos, Solitons and Fractals,* **12**, 2683.

Antoniou, I. and Prigogine, I. (1993). *Physica A* **192**, 443.

Baldo, M., Ferreira, L. S., and Streit, L. (1987). *Nuclear Physics A* **467**, 44.

Balslev, E. (1984). In *Resonances—Models and Phenomena,* S. Albeverio, L. Ferreira, and L. Streit, eds. Lecture Notes in Physics, Vol. 211, Springer-Verlag, Berlin, p. 27.

Baumgartel, H. (1976). *Math. Nachr.* **75**, 133 and references thereof.

Bohm, A. (1978). *Lett. Math. Phys.* **3**, 455.

Bohm, A. (1979). *Quantum Mechanics—Foundations and Applications,* Springer-Verlag, New York.

Bohm, A. (1981). *Journal of Mathematical Physics* **22**, 2813.

Bohm, A. (1994). *Quantum Mechanics—Foundations and Applications*, 3rd edn., Springer, New York.

Bohm, A. (1999). *Physical Review A* **60**, 861.

Bohm, A. (2003). *Fortshritte der Physik* **51**, 551–568, 569–603, 569–634.

Bohm, A., Loewe, M., Maxson, S., Patuleanu, P., Puntmann, C., and Gadella, M. (1997). *Journal of Mathematical Physics* **38**, 1.

Bohm, A., Maxson, S. Loewe, M., and Gadella, M. (1997). *Physica A* **236**, 485.

Bollini, C. G., Civitarese, O., De Paoli, A. L., and Rocca, M. C. (1996). *Journal of Mathematical Physics* **37**, 4235.

Brandas, E. (1986). The method of complex scaling, *International Journal of Quantum Chemistry* **S20**, 119.

Cohen-Tannoudji, C., Diu, B., and Laloe, F. (1977). *Quantum Mechanics,* Vol. II, Wiley, New York, pp. 1345, 1353–1354.

Duren, P. L. (1970). *Theory of H^p Spaces*, Academic Press, New York.
Erdély, A. *et al.* (1954). *Tables of Integral Transforms* (Bateman Manuscript Project), McGraw-Hill, New York, Vol. 1, Sect. 32, Eq. (3).
Fermi, E. (1932). *Reviews of Modern Physics* **4**, 87.
Fermi, E. (1950). *Nuclear Physics*, University of Chicago Press.
Feynman, R. P. (1948). *Reviews of Modern Physics* **20**, 369 [in particular pp. 372 and 379]
Ferreira, L. S. (1989). In Resonances, E. Brandas *et al.*, eds., Lecture Notes in Physics, Vol. 352, Springer-Verlag, Berlin, p. 201.
Gadella, M. (1983). *Journal of Mathematical Physics* **24**, 1462.
Gadella, M. (1997). *International Journal of Theoretical Physics* **36**, 2271.
Gamow, G. (1928). *Zeitschrift für Physik* **51**, 204.
Gell-Mann, M. and Hartle, J. B. (1994). In *Physical Origins of Time Asymmetry*, J. J. Halliwell, *et al.*, eds., Cambridge University, Cambridge.
Gell-Mann, M. and Hartle, J. B. (1995). University of California at Santa Barbara, Report No. UCSBTH-95-28, LANL Archives, gr-qc/9509054 and references thereof.
Goldberger, M. L. and Watson, K. M. (1964). *Collision Theory*, chap. 8, Wiley, New York.
Haag, R. (1990). *Communication in Mathematical Physics* **132**, 245.
Haag, R. (1997). Lectures at the Max Born Symposium on "Quantum Future," Przieka.
Hegerfeldt, G. C. (1994). *Physical Review Letters* **72**, 596.
Khalfin, L. A. (1957). *Soviet Physics-Doklady* **2**, 340–344.
Khalfin, L. A. (1958). *Soviet Physics-JETP* **6**, 1053–1063.
Kielanowski, P. (2003). Relativistic gamow vectors. *International Journal of Theoretical Physics* **42**, 2339–2355.
Kukulin, V. I., Krasnopolsky, V. M., and Horacek, J. (1989). *Theory of Resonances*, Kluwer Academic, Dordrecht.
Lee, T. D. (1981). *Particle Physics and Introduction to Field Theory*, chap. 13, Harwood Academic, New York.
Lee, T. D., Oehme, R., and Yang, C. N. (1957). *Physical Review* **105**, 340.
Levy, M. (1959). *Nuovo Cimento* **13**, 115.
Merzbacher, E. (1970). *Quantum Mechanics*, chap. 18, Wiley, New York.
Nathan, A. M., Garvey, G. T., Paul, P., and Warburton, E. K. (1975). *Physical Review Letters* **35**, 1137.
Newton, R. G. (1982). *Scattering Theory of Waves and Particles*, 2nd edn., Springer-Verlag, Berlin. [in particular chap. 7]
Particle Data Group (2002). *Physical Review D: Particles and Fields* **66**, 01001–01001.
Reed, M. and Simon, B. (1978). *Methods of Modern Mathematics III, IV*, Academic Press, New York.
Reinhardt, W. P. (1982). *Ann. Rev. Phys. Chem.* **33**, 223.
Siegert, A. F. (1939). *Physical Review* **56**, 750.
Simon, B. (1978). *International Journal of Quantum Chemistry* **14**, 529.
Simon, B. (1979). *Physics Letters* **71A**, 211.
Thomson, J. J. (1884). *Proc. London Math. Soc.* **15**, 197.
Tolstikhin, O. *et al.*, (1998). *Physical Review A* **58**, 207.
van Kampen, N. (2002). Proceedings of the XXI Solvay Conference 1999. *Advances in Chemical Physics* **122**, 301.
Vertse, T., *et al.*, (1989). In Resonances, E. Brandas *et al.*, eds., Lecture Notes in Physics, Vol. 352, Springer-Verlag, Berlin, p. 174.
von Brentano, P. (1996). *Physics Reports* **264**, 57, and references thereof.
Weinberg, S. (1995). *The Quantum Theory of Fields, Vol. 1*, Chap. 3, Cambridge University Press, Cambridge.
Weisskopf, V. and Wigner, E. P. (1930). *Zeitschrift für Physik* **63**, 54; **65**, 18.